International Journal of Theoretical Physics, Vol. 32, No. 10, 1993

Logicoalgebraic Structures I

Sylvia Pulmannová[1]

Received April 30, 1993

In this survey article, some basic properties of quantum logics are described, such as physical motivation, basic examples and constructions, and state spaces and observables. Some recent results concerning joint distributions and commutators and their relations to uncertainty relations, Bell inequalities, independence of center, automorphisms and state space, are included.

1. INTRODUCTION

The development of quantum mechanics has shown that the classical Kolmogorovian probability theory based on a triple (Ω, S, P), where Ω is a nonempty set, S is a σ-algebra of subsets of Ω, and P is a probability measure on S, is not suitable to describe quantum mechanical experiments. An alternative approach, the traditional quantum mechanics based on a Hilbert space, was suggested by von Neumann. The latter model has been successfully used to describe quantum mechanical experiments, but failed in a satisfactory physical motivation. In the historic paper of Birkhoff and von Neumann (1936), "The logic of quantum mechanics," the authors suggested to replace the Boolean σ-algebra B, representing the set of random events in the classical model, by a more general algebraic structure called a "quantum logic." The authors suggested a kind of modular lattice, a continuous geometry, as the suitable replacement. This algebraic structure is nondistributive, but contains many Boolean subalgebras, and the basic idea was that these Boolean subalgebras should represent those propositions (random events) which can be simultaneously verified by one experiment; in the traditional Hilbert space model, the role of "quantum logic" is played by the lattice of all closed subspaces of the underlying Hilbert space, which is not modular unless the Hilbert space is finite

[1]Mathematical Institute, Slovak Academy of Sciences, 814 73 Bratislava, Slovakia.

0020-7748/93/1000-1661$07.00/0

dimensional. But finite-dimensional Hilbert spaces are not sufficient to describe all quantum mechanical experiments. Therefore, in the further development, modularity has been replaced by a weaker property, orthomodularity, which is shared by both Boolean algebras and lattices of closed subspaces of Hilbert spaces of any dimension. This new approach started to be intensively developed in the 1960s in basic papers by, e.g., Zierler (1961), Varadarajan (1962), Mackey (1963), Piron (1964), MacLaren (1964), Gunson (1967), Gudder (1965), and Pool (1968).

In Mackey's (1963) work a probabilistic motivation of the quantum logic approach was given. We will briefly describe his axioms. Mackey starts with two abstract sets $\mathcal{O}$ and $\mathcal{S}$, where $\mathcal{O}$ represents the set of all observables (that is, physical quantities), and $\mathcal{S}$ represents the set of all physical states of a physical system. We suppose, as usual, that the result of a measurement of an observable is a real number. From any physical theory, we expect an answer to the question: "If a physical system is in the state $s \in \mathcal{S}$, what is the probability that a measurement of an observable $A \in \mathcal{O}$ gives a result in a subset E of the real line $\mathbb{R}$?" It is also natural to consider only Borel subsets of $\mathbb{R}$, and we denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra of $\mathbb{R}$. Mackey's axioms then can be written as follows:

1. There is a function $p: \mathcal{S} \times \mathcal{O} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ which for every fixed $s \in \mathcal{S}$ and $A \in \mathcal{O}$ is a probability measure on $\mathcal{B}(\mathbb{R})$.

2. $p(s_1, A, E) = p(s_2, A, E)$ for every $A \in \mathcal{O}$ and every $E \in \mathcal{B}(\mathbb{R}) \Rightarrow s_1 = s_2$.

3. $p(s, A_1, E) = p(s, A_2, E)$ for every $s \in \mathcal{S}$ and every $E \in \mathcal{B}(\mathbb{R}) \Rightarrow A_1 = A_2$.

Roughly speaking, axioms 2 and 3 mean that we can distinguish two states or two observables only by results of measurements. Define an equivalence relation on $\mathcal{O} \times \mathcal{B}(\mathbb{R})$ as follows: $(A, E) \sim (B, F)$ if $p(s, A, E) = p(s, B, F)$ for all $s \in \mathcal{S}$ and let $|A, E|$ be the equivalence class containing (A, E).

Put $L = \{|A, E| : A \in \mathcal{O}, E \in \mathcal{B}(\mathbb{R})\}$. For $s \in \mathcal{S}$ let $s: L \to [0, 1]$ be defined by $s(a) = p(s, A, E)$ when $a = |A, E|$.

4. For any $s_i \in \mathcal{S}$, $\alpha_i \geq 0$, $i \in \mathbb{N}$, $\sum_{i \in \mathbb{N}} \alpha_i = 1$ there is $s \in \mathcal{S}$ such that $s(a) = \sum_{i \in N} \alpha_i s_i(a) = s(a)$ $(a \in L)$.

5. If $(a_i)_{i \in \mathbb{N}} \subset L$ are such that $s(a_i) + s(a_j) \leq 1$ for any $i, j \in \mathbb{N}$, $i \neq j$, then there is a $b \in L$ such that $s(b) + \sum_{i \in \mathbb{N}} s(a_i) = 1$.

Axiom 4 defines a convex structure on $\mathcal{S}$. For $a, b \in L$, define $a \leq b$ if $s(a) \leq s(b)$ for every $s \in \mathcal{S}$, and put $a' = |A, \mathbb{R} \backslash E|$ if $a = |A, E|$. We can then derive, using axiom 5, that L becomes a σ-orthomodular poset and element of $\mathcal{S}$ define states (i.e., probability measures) on L (Mackey, 1963; Maczynski, 1967).

A σ-*orthomodular poset* (=a *quantum logic*) $L(\leq, 0, 1, ')$ $(0 \neq 1)$ is a partially ordered set L with 0 and 1 with an orthocomplementation $': L \to L$ such that:

(i) $(a')' = a$.

(ii) $a \leq b \Rightarrow b' \leq a'$.

(iii) $a \vee a' = 1$.

(iv) $(a_i)_{i \in \mathbb{N}} \subset L$, $a_i \leq a_j'$, whenever $i \neq j \Rightarrow \bigvee_{i \in \mathbb{N}} a_i \in L$.

(v) $a \leq b \Rightarrow b = a \vee (a' \wedge b)$ [where $a' \wedge b$ exists due to (i), (ii), and (iv)].

Here $\vee$ and $\wedge$ mean supremum and infimum, respectively, if they exist in L. A quantum logic L is a lattice if $a \vee b$, $a \wedge b$ exist in L for any $a, b \in L$ (Beltrametti and Cassinelli, 1981; Beran, 1984; Gudder, 1979; Kalmbach, 1983; Mackey, 1963; Piron, 1976; Pták and Pulmannová, 1991; Varadarajan, 1968/1970).

A *state* on L is a mapping $m: L \to [0, 1]$ such that

(i) $m(1) = 1$.

(ii) $(a_i)_{i \in \mathbb{N}}$, $a_i \leq a_j'$, for any i, j, $i \neq j \Rightarrow m(\bigvee_{i \in \mathbb{N}} a_i) = \sum_{i \in \mathbb{N}} m(a_i)$.

An *observable* on L is a mapping $x: \mathscr{B}(\mathbb{R}) \to L$ such that

(i) $x(\mathbb{R}) = 1$.

(ii) $x(E^c) = x(E)'$ (where $E^c = \mathbb{R} \backslash E$).

(iii) $x(\bigcup_{i \in \mathbb{N}} E_i) = \bigvee_{i \in \mathbb{N}} x(E_i)$ whenever $(E_i)_{i \in \mathbb{N}} \subset \mathscr{B}(\mathbb{R})$, $E_i \cap E_j = \emptyset$ for any i, j, $i \neq j$.

In the probabilistic interpretation, L is considered as the set of all random events of a quantum mechanical experiment, states are interpreted as probability measures, and observables as random variables (Beltrametti and Cassinelli, 1981; Gudder, 1979; Mackey, 1963; Pták and Pulmannová, 1991).

2. EXAMPLES OF LOGICS

2.1. Every Boolean σ-algebra is a quantum logic in a natural way. It can be easily seen that a quantum logic L is a Boolean σ-algebra if and only if L is a distributive lattice. By the Loomis (1947) Sikorski (1949) theorem, if B is a Boolean σ-algebra, then there is a measurable space $(\Omega, \mathscr{S})$ (where $\mathscr{S}$ is a σ-algebra of subsets of a set Ω) and there is a σ-homomorphism h from $\mathscr{S}$ onto B. Moreover, by Varadarajan (1962, 1968/1970), to every observable $x: \mathscr{B}(\mathbb{R}) \to B$ there is a $(\mathscr{S}, \mathscr{B}(\mathbb{R}))$-measurable function $f: \Omega \to \mathbb{R}$ such that $x = h \cdot f^{-1}$. If m is a state on B, then $m \cdot h$ is a probability measure on $\mathscr{S}$. We see that in this case we obtain the usual Kolmogorovian model $(\Omega, \mathscr{S}, m \cdot h)$.

Let L be a quantum logic, x an observable on L, $f: \mathbb{R} \to \mathbb{R}$ a Borel function, and m a state on L. Then the range $x(\mathscr{B}(\mathbb{R}))$ of x is a Boolean sub-σ-algebra of L, $x \cdot f^{-1} \equiv f(x)$ is an observable, and $m \cdot x \equiv m_x$ is a probability measure on $\mathscr{B}(\mathbb{R})$, which is called the *probability distribution* of x in the state m. The *expectation* of x in m is defined by

$$m(x) = \int_{\mathbb{R}} t\, m_x(dt)$$

(if the integral exists). The *variation* of x in m is defined by

$$\mathrm{var}_m(x) = \int [t - m(x)]^2\, m_x(dt)$$

(if the integral exists and is finite).

If $x_1, \ldots, x_n$ are observables on a quantum logic L such that $\bigcup_{i \le n} x_i(\mathscr{B}(\mathbb{R})) \subset B$, where B is a Boolean sub-σ-algebra of L, we find a functional representation $x_i = h \cdot f_i^{-1}$, $i = 1, 2, \ldots, n$, where h is the Loomis–Sikorski σ-homomorphism for B. This enables us to define a *functional calculus* for these observables (Pták and Pulmannová, 1991; Varadarajan, 1968/1970).

2.2. Let H be a Hilbert space (real or complex). The set $L(H)$ of all closed linear subspaces of H with the partial order defined by inclusion and orthocomplementation $M \to M'$ defined by $M' = \{\phi \in H : \langle \phi, \psi \rangle = 0$ for $\forall \psi \in M\}$, $M \in L(H)$, is a complete orthomodular lattice, i.e., a quantum logic. By the spectral theorem, observables correspond to self-adjoint operators. If $2 < \dim H \le \aleph_0$, then, by Gleason's theorem, every state is of the form

$$m(M) = \mathrm{tr}\, TP^M$$

where P^M is the orthogonal projection onto M, and T is a positive self-adjoint operator with trace $\mathrm{tr}\, T = 1$ (e.g., Gleason, 1957; Dvurečenskij, 1992; Pták and Pulmannová, 1991; Varadarajan, 1968/1970).

2.3. The projection lattice $L(\mathscr{A})$ of every von Neumann algebra $\mathscr{A}$ is a quantum logic (Kalmbach, 1983; Varadarajan, 1968/1970).

2.4. Let X be a nonempty set. Let Q be a family of subsets of X such that:

(i) $X \in Q$.

(ii) $A \in Q \Rightarrow X \backslash A \in Q$.

(iii) $(A_i)_{i \in \mathbb{N}} \subset Q$, $A_i \cup A_j = \varnothing$ for $i \neq j \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in Q$.

Then Q is a so-called *concrete logic*. It was introduced by Suppes (1966). Observables on Q are mappings f^{-1}: $\mathscr{B}(\mathbb{R}) \to Q$, where f: $X \to \mathbb{R}$ is a Q-measurable function (e.g., Gudder, 1979; Pták and Pulmanová, 1991).

2.5. There are several constructions that enable us to construct new logics from given ones (Greechie, 1971; Kalmbach, 1983; Pták and Pulmannová, 1991).

(a) For $a \in L$, put $L_{[0,a]} = \{b \in L : b \leq a\}$. Then $L_{[0,a]}$ is a quantum logic with the relative orthocomplementation $b'^{a} = b' \wedge a$, $b \in L_{[0,a]}$.

(b) Let L_1, L_2 be quantum logics. Put $L = L_1 \times L_2$, and define $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$; $(a, b)' = (a', b')$. Then L is a quantum logic, the *direct product* of L_1 and L_2.

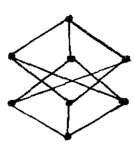
(c) Let L_1, L_2 be quantum logics. Put $L = L_1 \cup L_2$, identify ("past") the zero element 0_1 in L_1 with the zero element 0_2 in L_2 and, similarly, the unit element 1_1 with 1_2. Then define the partial order on $L_1 \cup L_2$ as the union of the partial orders $\leq_1$ on L_1 and $\leq_2$ on L_2. Then L is a quantum logic, the *horizontal sum* or $\{0, 1\}$*-pasting* of L_1 and L_2.

(d) *Greechie constructions* are $\{0, 1\}$-pastings of special atomic Boolean algebras additionally pasted in some atoms (Greechie, 1971; Kalmbach, 1983; Pták and Pulmannová, 1991). Greechie constructions can be represented by *Greechie diagrams*. A smooth line in a Greechie diagram of a logic L represents a Boolean algebra (a *block* of L), points represent atoms, and angles represent pastings (see Figures 1 and 2).

Greechie found examples of quantum logics that have no states. The following example is a simplified version by Rogalewicz (Pták and Pulmannová, 1991). The quantum logic with the Greechie diagram of Figure 3 can be covered either by 12 disjoint blocks or by 13 disjoint blocks. Since for every state the sum of its values on all atoms of every block is 1, if a state exists, we have $12 = 13$, a contradiction.

We note that a direct product of a logic L_1 with no state with $L_2 = \{0, 1\}$ gives a logic with exactly one state (see Pták and Pulmannová, 1991).

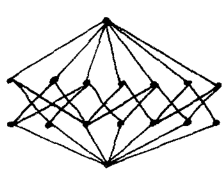
Let $\mathcal{S}(L)$ denote the set of all states on L, the *state space* of L. In the above examples we have seen that the state space of a logic L can be very poor, even empty. From the physical point of view, quantum logics with an



Boolean algebra with 3 atoms

Greechie diagram

Fig. 1.



Dilworth's lattice

Greechie diagram

Fig. 2.

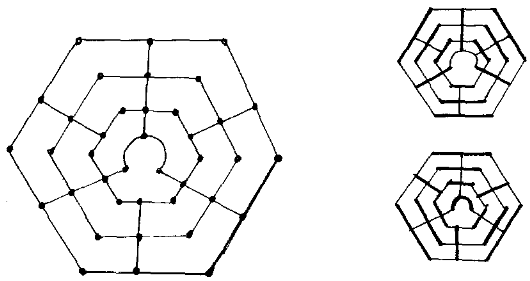


Fig. 3.

ample supply of states are of interest. Let us consider the following definitions. A subset $S \subset \mathcal{S}(L)$ is:

(a) *rich* if $a, b \in L$, $a \nleq b \Rightarrow \exists s \in S$: $s(a) = 1$, $s(b) < 1$;

(b) *full* if $a, b \in L$, $a \nleq b \Rightarrow \exists s \in S$: $s(a) > s(b)$;

(c) *unital* if $a \in L$, $a \neq 0 \Rightarrow \exists s \in S$: $s(a) = 1$.

It is easy to see that rich $\Rightarrow$ full and unital. All σ-algebras of sets, concrete logics, and Hilbert-space logics $L(H)$ are examples of quantum logics with rich state spaces.

3. COMPATIBILITY

From the physical point of view, it is very important to characterize those subsets of a quantum logic L which can be embedded into Boolean sub-σ-algebras of L. Elements of such subsets correspond to random events which can be simultaneously experimentally verified by the same experimental arrangement. The following definitions and results can be found in Pták and Pulmannová (1991).

A subset B of a quantum logic L is a *Boolean sub-σ-algebra* of L if (i) $0 \in B$, (ii) $a \in B \Rightarrow a' \in B$, (iii) $(a_i)_{i \in \mathbb{N}} \subset B$ and $a_i \leq a_j'$ whenever $i \neq j \Rightarrow \bigvee_{i \in \mathbb{N}} a_i \in B$, and (iv) $a, b, c \in B \Rightarrow a \vee b$ exists in L and belongs to B, and $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$. A *block* of L is a maximal Boolean sub-σ-algebra of L. Due to Zorn's lemma, every Boolean sub-σ-algebra of L is contained in a block.

A subset $A \subset L$ is called *compatible*, or elements of A are called *compatible elements*, if there is a Boolean sub-σ-algebra of L containing A.

If $A = \{a, b\}$, then A is compatible if and only if $a = (a \wedge b) \vee (a \wedge b')$ and $b = (a \wedge b) \vee (a' \wedge b)$ (in the sense that all the infima exist and the equality holds). If $\{a, b\}$ is compatible, we write $a \leftrightarrow b$. If L is a lattice, then A is compatible if and only if $a \leftrightarrow b$ for every $a, b \in A$ (Varadarajan, 1968/1970; Kalmbach, 1983; Pták and Pulmannová, 1991).

Let L be any logic and let $F=\{a_1,\ldots,a_n\}$. Define $a^0=a'$, $a^1=a(a\in L)$. If the elements $\bigwedge_{i=1}^{n} a_i^{f(i)}$, $f\in\{0,1\}^n$ all exist, we call the element

$$\operatorname{com} F=\bigvee_{f\in\{0,1\}^n}\bigwedge_{i=1}^{n} a_i^{f(i)}$$

the *commutator* of the set F (Marsden, 1970; Beran, 1984; Kalmbach, 1983; Pták and Pulmannová, 1991). For example,

$$\operatorname{com}\{a,b\}=(a\wedge b)\vee(a'\wedge b)\vee(a\wedge b')\vee(a'\wedge b')$$

Proposition 1 (Pták and Pulmannová, 1991). A subset A of a logic L is compatible if and only if the element com F exists for every finite set $F\subset A$, and com $F=1$.

Let A be any subset of L. If the element

$$\operatorname{com} A=\bigwedge\{\operatorname{com} F: F\subset A, F \text{ is finite}\}$$

exists, we will call it the *commutator* of A. If, for example, L is a lattice, then it is a σ-lattice, and hence the commutator exists for every at most countable subset of L.

Proposition 2 (Pták and Pulmannová, 1991). Let L be a quantum logic which is a lattice. If $c=\operatorname{com} A$ exists for a subset A of L, then:

(i) $c\leftrightarrow a$ for any $a\in A$.

(ii) $\{c\wedge a: a\in A\}$ is a compatible set.

(iii) If $d\in L$ has the properties (i) and (ii), then $d\leq c$.

Let L be a lattice. Let $\{x_i: i\in I\}$ be a system of observables on L. Define

$$\operatorname{com}\{x_i: i\in I\}=\operatorname{com}\left(\bigcup_{i\in I} x_i(\mathscr{B}(\mathbb{R}))\right)$$

Proposition 3 (Pulmannová and Dvurečenskij, 1985; Pták and Pulmannová, 1991). Let $\{x_i: i\in I\}$ be a system of observables on a quantum logic L which is a lattice. If I is a countable set, then $c=\operatorname{com}\{x_i: i\in I\}$ exists. Moreover, the mappings $x_i^c:\mathscr{B}(\mathbb{R})\to L_{[0,c]}$, defined by $x_i^c(E)=x_i(E)\wedge c$, $i\in I$, are compatible observables on $L_{[0,c]}$.

4. JOINT DISTRIBUTIONS OF OBSERVABLES OF GUDDER TYPE

In what follows, we will assume that L is a quantum logic which is a lattice. Let $x_1,\ldots,x_n$ be observables and let m be a state on L. For

$E_1 \times \cdots \times E_n \in \mathscr{B}(\mathbb{R}^n)$ put

$$\mu(E_1 \times \cdots \times E_n) = m(x_1(E_1) \wedge \cdots \wedge x_n(E_n))$$

If μ can be extended to a measure on $\mathscr{B}(\mathbb{R}^n)$, we will call μ the *joint distribution of Gudder type* of the observables $x_1, \ldots, x_n$ in the state m (Gudder, 1968, 1979; Dvurečenskij, 1992; Dvurečenskij and Pulmannová, 1984; Pták and Pulmannová, 1991). For another type of joint distribution, the so-called Urbanik type, see Gudder, 1968, 1979; Pták and Pulmannová, 1991; Urbanik, 1961; Varadarajan (1968/1970). In particular, for one observable x, $\mu = m_x$. If the joint distribution exists, we can define a joint distribution function F: $\mathbb{R}^n \to [0, 1]$ by

$$F(t_1, \ldots, t_n) = m(x_1(-\infty, t_1] \wedge \cdots \wedge x_n(-\infty, t])$$

and a joint characteristic function Φ: $\mathbb{R}^n \to \mathbb{C}$ by

$$\Phi(u_1, \ldots, u_n) = \int_{\mathbb{R}^n} \exp\left(i \sum_{j=1}^{n} u_j t_j\right) dF(t_1, \ldots, t_n)$$

Theorem 4 (Pulmannová and Dvurečenskij, 1985; Pták and Pulmannová, 1991). The joint distribution of Gudder type of the observables $x_1, \ldots, x_n$ in the state m exists if and only if $m(\mathrm{com}\{x_1, \ldots, x_n\}) = 1$.

In particular, for compatible observables the joint distribution exists in every state. On the other hand, if the joint distribution exists in every state in a full or unital set of states, then the observables are compatible.

Corollary. If the joint distribution of Gudder type of the observables $x_1, \ldots, x_n$ exists in the state m, then the reduced compatible observables $x_1^c, \ldots, x_n^c$ on the logic $L_{[0,c]}$, where $c = \mathrm{com}\{x_1, \ldots, x_n\}$, have the same probability distribution in the state $m \mid L_{[0,c]}$ as the original observables in the state m.

The above corollary describes the so-called "compatible reduction" of observables.

Let $L = L(H)$ be a Hilbert space logic. Let $x_1, \ldots, x_n$ be observables on L (i.e., spectral measures) corresponding to self-adjoint operators $A_1, \ldots, A_n$. Then $\mathrm{com}\{x_1, \ldots, x_n\}$ can be alternatively defined as

$$\begin{aligned}\mathrm{Com}\{x_1, \ldots, x_n\} = \{\phi \in H : P^{x_1(E_1)} \ldots P^{x_n(E_n)}\phi \\ = P^{x_{\pi(1)}(E_{\pi(1)})} \ldots P^{x_{\pi(n)}(E_{\pi(n)})}\phi \\ \text{for every permutation } \pi \text{ of } \{1, \ldots, n\}\}\end{aligned}$$

The set $\mathrm{Com}\{x_1, \ldots, x_n\}$ is a closed subspace of H which is invariant under $x_i(E)$, $i = 1, \ldots, n$, $E \in \mathscr{B}(\mathbb{R})$. The observables $x_1, \ldots, x_n$ have the joint distribution in the state $m = \sum w_j P_{\phi_j}$ if and only if $\phi_j \in$

$\mathrm{Com}\{x_1, \ldots, x_n\}$ for all $j \leq n$. The compatible reduction is then the reduction of the self-adjoint operators $A_1, \ldots, A_n$ to the invariant subspace $\mathrm{Com}\{x_1, \ldots, x_n\}$, where the reduced operators commute (Dvurečenskij, 1992; Pták and Pulmannová, 1991).

5. UNCERTAINTY RELATIONS

Let L be a quantum logic, which is a lattice. Assume that there is a unital set $\mathscr{S}$ of states on L. For an observable x on L define

$$V(x) = \{m \in \mathscr{S} : \mathrm{var}_m(x) < \infty\}$$

Let $x_1, \ldots, x_n$ be observables. Two cases can occur.

$$\text{(A)}\quad (\exists \epsilon > 0)\left(\forall m \in \bigcap_{i \leq n} V(x_i)\right): \mathrm{var}_m(x_1) \ldots \mathrm{var}_m(x_n) \geq \epsilon$$

$$\text{(B)}\quad (\forall \epsilon > 0)\left(\exists m \in \bigcap_{i \leq n} V(x_i)\right): \mathrm{var}_m(x_1) \ldots \mathrm{var}_m(x_n) < \epsilon$$

If (A) occurs, we say that the observables $x_1, \ldots, x_n$ satisfy the *uncertainty relation* (Lahti, 1980; Pták and Pulmannová, 1991).

If (A) is satisfied, then at least two of $x_1, \ldots, x_n$ must be unbounded [recall that an observable x is bounded if there is a compact set $C \subset \mathbb{R}$ such that $x(C) = 1$].

Theorem 5 (Pulmannová and Dvurečenskij, 1985; Pták and Pulmannová, 1991). If the uncertainty relation (A) is satisfied by the observables $x_1, \ldots, x_n$, then

$$\mathrm{com}\{x_1, \ldots, x_n\} = 0$$

Consequently, $x_1, \ldots, x_n$ do not have the joint distribution of Gudder type in any state $m \in \mathscr{S}$. A well-known example of a pair of observables satisfying the uncertainty relation are the position and momentum operators on $L_2(\mathbb{R})$.

6. A CLASSIFICATION OF QUANTUM LOGICS

Let L be a quantum logic and let $\mathscr{S}(L)$ denote the set of all states on L (i.e., the state space of L). Recall that a state m on L is called a $\{0, 1\}$-state if $m(a) \in \{0, 1\}$ for every $a \in L$. According to Gudder (1979), a quantum logic L is concrete iff it has a full set of $\{0, 1\}$-states.

We will introduce four classes of unital logics denoted by $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, and $\mathscr{L}_4$ ($a \not\leftrightarrow b$ means "a and b are not compatible"):

$$L \in \mathscr{L}_1 \Leftrightarrow (a, b \in L, a \nleftrightarrow b \Rightarrow \exists s \in \mathscr{S}(L) : s(a) = 1, s(b) \neq 1)$$

$$L \in \mathscr{L}_2 \Leftrightarrow (a, b \in L, a \nleftrightarrow b \Rightarrow \forall \epsilon > 0 \, \exists s \in \mathscr{S}(L) : s(a) = 1, s(b) \geq 1 - \epsilon)$$

$$L \in \mathscr{L}_3 \Leftrightarrow (a, b \in L, a \nleftrightarrow b \Rightarrow s \in \mathscr{S}(L), s(a) = 1, s(b) = 1)$$

$$L \in \mathscr{L}_4 \Leftrightarrow (a, b \in L, a \nleftrightarrow b \Rightarrow \exists \{0, 1\}\text{-state } s \in \mathscr{S}(L), s(a) = 1, s(b) = 1$$
$$\text{and the set of } \{0, 1\}\text{-states is unital})$$

Theorem 6 (Pták and Rogalewicz, 1983; and Pták and Pulmannová, 1991). The following inclusions hold:

$$\mathscr{L}_4 \subset \mathscr{L}_3 \subset \mathscr{L}_2 \subset \mathscr{L}_1$$

and all these inclusions are proper. Moreover, $\mathscr{L}_1$ is the class of rich logics, and L_4 is the class of concrete logics.

The class $\mathscr{L}_1$ of rich logics contains the Hilbert space logic $L(H)$ of a Hilbert space H. Therefore, there are couples of observables on quantum logics in the class $\mathscr{L}_1$ which satisfy the uncertainty relations. The following theorem asserts that in $\mathscr{L}_3$ (and hence also in $\mathscr{L}_4$) uncertainty relations cannot occur for any observables.

Theorem 7 (Pulmannová, 1988). If L belongs to $\mathscr{L}_3$, then the uncertainty relations cannot be satisfied by any pair of observables on L.

It is not known whether the uncertainty relations can be satisfied by observables on a logic in the class $\mathscr{L}_2$.

7. BOOLEAN QUOTIENTS, BELL INEQUALITIES, AND JOINT DISTRIBUTIONS OF OBSERVABLES

Let L be an orthomodular lattice. Recall that a subset J of L is a *p-ideal* if (i) $a \in J$, $b \in L$, $b \leq a \Rightarrow b \in J$, (ii) $a, b \in J \Rightarrow a \vee b \in J$, (iii) $a \in J$, $b \in L \Rightarrow (a \vee b') \wedge b \in J$. A *congruence* of L is an equivalence relation θ such that $a\theta b \Rightarrow a'\theta b'$ and $a_1 \theta b_1$, $a_2 \theta b_2 \Rightarrow a_1 \vee a_2 \theta b_1 \vee b_2$ and $a_1 \wedge a_2 \theta b_1 \wedge b_2$. If θ is a congruence, the set of all congruence classes forms an orthomodular lattice called the quotient of L. If θ is a congruence, then the equivalence class $\theta(0)$ containing 0 is a p-ideal. Conversely, if J is a p-ideal, then the relation θ defined by $a\theta b$ iff $(a \vee b) \wedge (a' \vee b') \in J$ is a congruence and $\theta(0) = J$. If θ is a congruence and J is the corresponding p-ideal, we denote the corresponding quotient by L/θ, or equivalently by L/J (Beran, 1984; Kalmbach, 1983; Marsden, 1970).

It was proved by Marsden (1970) that the quotient L/J is a Boolean algebra iff $J \supset J_c$, where J_c denotes the smallest p-ideal of L containing all elements $\text{com}(a, b)' = (a \vee b) \wedge (a' \vee b) \wedge (a \vee b') \wedge (a' \vee b')$, $a, b \in L$.

If L is an orthomodular σ-lattice, we can introduce the notions of a σ-p-ideal, σ-congruence, and σ-quotient in a natural way. For example, a p-ideal J is a σ-p-ideal provided that $(a_i)_{i\in\mathbb{N}} \subset J \Rightarrow \bigvee_{i\in\mathbb{N}} a_i \in J$.

The famous Bell inequalities may be considered as "tests of the existence of hidden variables." According to the hidden variables hypothesis, there is a classical description "hidden" behind the quantum mechanical description of a physical system (Beltrametti and Cassinelli, 1981; Beltrametti and Maczynski, 1991; Gudder, 1979; Pitowski, 1989; Pulmanová and Majernik, 1992). Recently, the following types of Bell inequalities appeared in the context of quantum logics (L is an orthomodular lattice and s is a state on L): the Bell–Wigner-type inequalities

$$s(a) + s(b) + s(c) - s(a \wedge b) - s(a \wedge c) - s(b \wedge c) \leq 1 \qquad \text{(BW)}$$

($a, b, c \in L$), and the Clauser–Horne-type inequalities

$$1 \geq s(b) + s(c) - s(a \wedge b) - s(b \wedge c) - s(c \wedge d) + s(a \wedge c) \geq 0 \quad \text{(CH)}$$

($a, b, c, d \in L$) (Beltrametti and Maczynski, 1991; Pitowski, 1989).

Theorem 8 (Pulmannová and Majernik, 1992). Let L be an orthomodular σ-lattice and s be a state on L. The following statements are equivalent:

(i) There is a σ-p-ideal J such that the quotient L/J is a Boolean σ-algebra and $s(a) = 0$ for all $a \in J$.

(ii) There is a Boolean σ-algebra B, a surjective σ-homomorphism $\phi: L \to B$, and a state $\bar{s}$ on B such that $\bar{s} \cdot \phi = s$.

(iii) Inequalities (BW) are satisfied on (L, s).

(iv) Inequalities (CH) are satisfied on (L, s).

(v) The set X of all observables on L has a joint distribution in the state s.

8. INDEPENDENCE OF CENTER, AUTOMORPHISM GROUP, AND STATE SPACE

In the preceding section, we have seen several equivalent conditions on quantum logics. In the present section we show that such important characterizations as center, automorphism group, and state space are completely independent.

We will consider an orthomodular poset L, that is, L is a partially ordered set with an orthocomplementation $': L \to L$ such that (i) $a'' = a$, (ii) $a \leq b \Rightarrow b' \leq a'$, (iii) $a \vee a' = 1$, (iv) $a \leq b' \Rightarrow a \vee b$ exists in L, and (v) $a \leq b \Rightarrow b = a \vee (a' \wedge b)$. By a state on L we will mean a finitely additive state m (that is, $m: L \to [0, 1]$ such that $m(1) = 1$ and $m(a \vee b) = m(a) + m(b)$ whenever $a \leq b'$). The state space of L is the set of all finitely additive states on L. An *automorphism* of L is bijection $\alpha: L \to L$ such that both α and α^{-1} preserve the orthocomplementation and the partial ordering. The *center* of L is the intersection of all blocks of L. The following theorem is the result of investigations carried out by Greechie (1971), Schrag (1976), Kalmbach (1984), Kallus and Trnková (1987), Navara *et al.* (1988), and Navara (1992).

Theorem 9 (Navara, 1992). Suppose that K is a logic (orthomodular poset) admitting at least one state, G is a group, C is a compact convex subset of a locally convex topological linear space, and B is a Boolean algebra.

Then there is a logic L such that K is a sublogic of L, the group of automorphisms of L is isomorphic to G, the state space of L is affinely homeomorphic to C, and the center of L is Boolean isomorphic to B.

REFERENCES

Beltrametti, E., and Cassinelli, G. (1981). *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Beltrametti, E., and Maczynski, M. (1991). On a characterization of classical and non-classical probabilities, *Journal of Mathematical Physics*, **32**, 1280–1290.

Beran, L. (1984). *Orthomodular Lattices, Algebraic Approach*, Academia, Prague, and Reidel, Dordrecht.

Birkhoff, G., and von Neumann, J. (1963). The logic of quantum mechanics, *Annals of Mathematics*, **37**, 823–843.

Dvurečenskij, A. (1992). *Gleason's Theorem and Its Applications*, Ister, Bratislava and Kluwer, Dordrecht.

Dvurečenskij, A., and Pulmannová, S. (1984). Connection between joint distributions and compatibility of observables, *Reports on Mathematical Physics*, **19**, 349–359.

Greechie, R. (1971). Orthomodular lattices admitting no states, *Journal of Combinatorial Theory*, **10**, 119–132.

Gleason, A. (1957). Measures on the closed subspaces of a Hilbert space, *Journal of Mathematics*, **6**, 885–894.

Gudder, S. (1965). Spectral methods for a generalized probability theory, *Transactions of the American Mathematical Society*, **119**, 428–442.

Gudder, S. (1968). Joint distributions of observables, *Journal of Mathematics and Mechanics*, **18**, 325–335.

Gudder, S. (1979). *Stochastic Methods in Quantum Mechanics*, Elsevier North-Holland, Amsterdam.

Gunson, J. (1967). On the algebraic structure of quantum mechanics, *Communications in Mathematical Physics*, **6**, 262–285.

Kallus, M., and Trnková, V. (1987). Symmetries and retracts of quantum logics, *International Journal of Theoretical Physics*, **26**, 1–5.

Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, New York.

Kalmbach, G. (1984). Automorphism groups of orthomodular lattices, *Bulletin of the Australian Mathematical Society*, **29**, 309–313.

Lahti, P. (1980). Uncertainty and complementarity in axiomatic quantum mechanics, *International Journal of Theoretical Physics*, **19**, 789–842.

Loomis, L. (1947). On the representation of σ-complete Boolean algebras, *Bulletin of the American Mathematical Society*, **53**, 757–760.

Mackey, G. (1963). *The Mathematical Foundations of Quantum Mechanics*, Benjamin, New York.

MacLaren, M. (1964). Atomic orthocomplemented lattices, *Pacific Journal of Mathematics*, **14**, 597–612.

Maczynski, M. (1967). A remark on Mackey's axiom system for quantum mechanics, *Bulletin de l'Academie Polonaise des Sciences*, **15**, 583–587.

Marsden, E. (1970). The commutator and solvability in a generalized orthomodular lattice, *Pacific Journal of Mathematics*, **33**, 357–361.

Navara, M. (1992). Independence of automorphism group, center and state space of quantum logics, *International Journal of Theoretical Physics*, **31**, 925–935.

Navara, M., Pták, P., and Rogalewicz, V. (1988). Enlargements of quantum logics, *Pacific Journal of Mathematics*, **35**, 361–369.

Pitowski, I. (1989). *Quantum Probability–Quantum Logic*, Springer, New York.

Piron, C. (1964). Axiomatique quantique, *Helvetica Physica Acta*, **37**, 439–468.

Piron, C. (1976). *Foundations of Quantum Physics*, Benjamin, Reading, Massachusetts.

Pool, J. (1968). Baer*-semigroups and the logic of quantum mechanics, *Communications in Mathematical Physics*, **9**, 118–141.

Pták, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.

Pták, P., and Rogalewicz, V. (1983). Measures on orthomodular partially ordered sets, *Journal of Pure and Applied Algebra*, **28**, 75–80.

Pulmannová, S. (1988). Uncertainty relations and state spaces, *Annales de l'Institut Henri Poincaré A*, **84**, 325–332.

Pulmannová, S., and Dvurečenskij, A. (1985). Uncertainty principle and joint distributions of observables, *Annales de l'Institut Henri Poincaré A*, **42**, 253–265.

Pulmannová, S., and Majerník, V. (1992). Bell inequalities on quantum logics, *Journal of Mathematical Physics*, **33**, 2173–2178.

Schrag, G. (1976). Every finite group is the automorphism group of some finite orthomodular lattice, *Proceedings of the American Mathematical Society*, **55**, 243–249.

Sikorski, R. (1949). On inducing of homomorphisms by mappings, *Fundamenta Mathematicae*, **36**, 7–22.

Suppes, P. (1966). The probabilistic argument for a nonclassical logic for quantum mechanics, *Philosophy of Sciences*, **33**, 14–21.

Urbanik, K. (1961). Joint probability distributions of observables in quantum mechanics, *Studia Mathematica*, **21**, 117–133.

Varadarajan, V. (1962). Probability in physics and a theorem on simultaneous observability, *Communications in Pure and Applied Mathematics*, **15**, 189–217.

Varadarajan, V. (1968/1970). *Geometry of Quantum Theory* I, II, van Nostrand, Princeton, New Jersey.

Zierler, N. (1961). Axioms for non-relativistic quantum mechanics, *Pacific Journal of Mathematics*, **11**, 1151–1169.